We will learn two ways to multiply vectors:

dot product and **cross product**

Name	Spits out	What it does
Dot product $\vec{u}\cdot\vec{v}$	$\vec{u}\cdot\vec{v}$ is a **scalar**	Measures how **parallel** the two vectors are
Cross product $\vec{u}\times\vec{v}$	$\vec{u}\times\vec{v}$ is a **vector**	Measures how **orthogonal** the two vectors are

Dot product has a memorable formula.

In 2D, $\vec{u}=\langle a_1, a_2\rangle$ $\vec{v}=\langle b_1, b_2\rangle$ then

$$\vec{u}\cdot\vec{v}=a_1b_1+a_2b_2.$$

In 3D, $\vec{u}=\langle a_1, a_2, a_3\rangle$ $\vec{v}=\langle b_1, b_2, b_3\rangle$ then

$$\vec{u}\cdot\vec{v}=a_1b_1+a_2b_2+a_3b_3$$

<u>Example</u>

- $\langle 3,2,1\rangle\cdot\langle -2,3,2\rangle=-6+6+2=2$
- $\langle 4,1,-2\rangle\cdot\langle 1,1,2\rangle=4+1-4=1$

In particular, if $\vec{u} = \langle a_1, a_2, a_3 \rangle$ and consider

$\vec{u} \cdot \vec{u}$, it becomes

$$\vec{u} \cdot \vec{u} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= a_1 \cdot a_1 + a_2 \cdot a_2 + a_3 \cdot a_3$$

$$= a_1^2 + a_2^2 + a_3^2 = |\vec{u}|^2.$$

So dot product can be used to obtain the length of a vector.

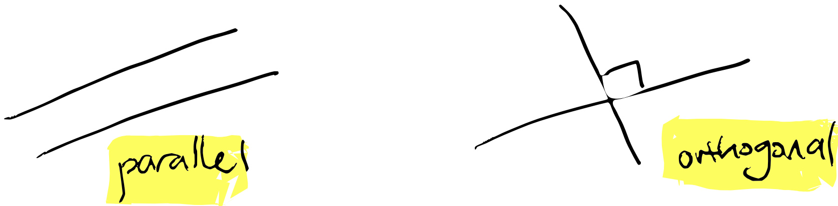
<u>Some algebra properties</u> (Easy if you write out the formula the above)

$\vec{u}, \vec{v}, \vec{w}$ vectors, a scalar.

- $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $\vec{u} \cdot \vec{0} = 0$
- $\vec{u} \cdot \vec{i}$ is the x-component of $\vec{u}$,
  $\vec{u} \cdot \vec{j}$ is the y-component of $\vec{u}$,
  $\vec{u} \cdot \vec{k}$ is the z-component of $\vec{u}$.
- $(a\vec{u}) \cdot \vec{v} = a(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (a\vec{v})$

Parallelism and orthogonality

We know intuitively what it means for a pair of lines to be parallel or orthogonal to each other.



In fact, dot product can precisely detect these two notions!

Two vectors $\vec{u}$ and $\vec{v}$ are

- parallel to each other if $|\vec{u}\cdot\vec{v}| = |\vec{u}|\cdot|\vec{v}|$.
- orthogonal to each other if $\vec{u}\cdot\vec{v} = 0$.

Example

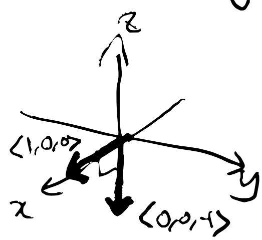
- We know $\vec{u}$ and $-3\vec{u}$ are parallel to each other. Indeed

  $\vec{u}\cdot(-3\vec{u}) = -3(\vec{u}\cdot\vec{u}) = -3|\vec{u}|^2$, so

  $|\vec{u}\cdot(-3\vec{u})| = 3|\vec{u}|^2 = |\vec{u}|\cdot|-3\vec{u}|$.

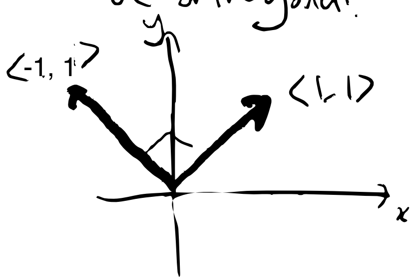
Example, cont'd

- We know $\langle 1,0,0\rangle$ and $\langle 0,0,-1\rangle$ should be orthogonal.



We check $\langle 1,0,0\rangle \cdot \langle 0,0,-1\rangle = 1\cdot 0 + 0\cdot 0 + 0\cdot(-1)$
$= 0.$

- We know $\langle 1,1\rangle$ and $\langle -1,1\rangle$ should be orthogonal.



We check $\langle 1,1\rangle \cdot \langle -1,1\rangle = 1 - 1 = 0$

Two vectors are parallel when the angle between them is either $0°$ or $180°$.
(0 or π in radians)

Two vectors are orthogonal when the angle between them is $90°$ ($\frac{\pi}{2}$ in radians)

More generally, dot product can measure the angle between two vectors.

<u>Theorem</u> If the angle between two vectors $\vec{u}$ and $\vec{v}$ is θ (in radians), then

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta.$$

<u>Examples</u>

Indeed, if $\theta = \frac{\pi}{2}$ (or $90°$), $\cos\theta = 0$, so

$$\vec{u}\cdot\vec{v} = 0$$

If $\theta = 0$, $\cos\theta = 1$, so

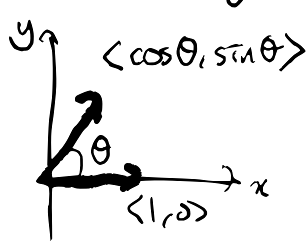
$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$$

If $\theta = \pi$ (or $180°$), $\cos\theta = -1$, so

$$\vec{u}\cdot\vec{v} = -|\vec{u}||\vec{v}|$$

Note that in both cases $\vec{u}$ and $\vec{v}$ are parallel, since $|\vec{u}\cdot\vec{v}| = |\vec{u}||\vec{v}|$.

Slightly more complicated: the vector $\langle\cos\theta, \sin\theta\rangle$ form an angle θ with $\langle 1, 0\rangle$.



We check

$$\langle\cos\theta, \sin\theta\rangle\cdot\langle 1, 0\rangle = \cos\theta$$

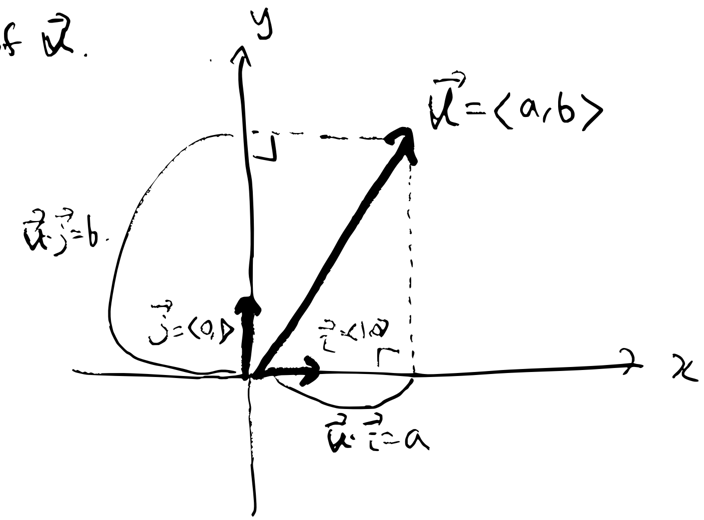
$$|\langle\cos\theta, \sin\theta\rangle| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

$$|\langle 1, 0\rangle| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1.$$

So $\langle\cos\theta, \sin\theta\rangle\cdot\langle 1, 0\rangle = \cos\theta \cdot |\langle\cos\theta, \sin\theta\rangle|\,|\langle 1, 0\rangle|.$

Now we learn how to take the projection of a vector onto another vector.

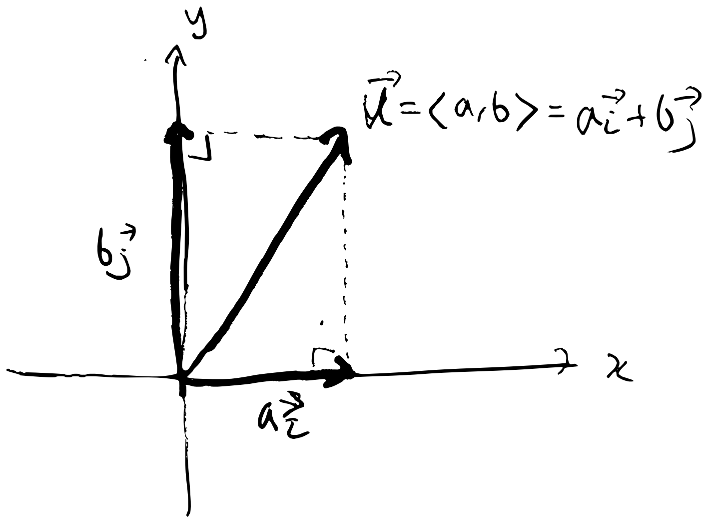
Remember that $\vec{u}\cdot\vec{i}$ picks the x-component of $\vec{u}$.



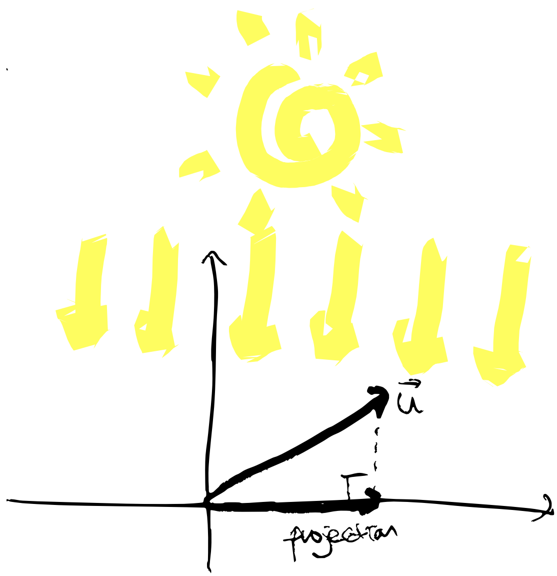
In general, for two vectors $\vec{u}$ and $\vec{v}$, the $\vec{v}$-direction component of $\vec{u}$ is given by

$$\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}, \text{ denoted as } \mathrm{comp}_{\vec{v}}\vec{u}.$$

You can furthermore decompose $\vec{u}$ into the sum of a $\vec{i}$-parallel vector and a $\vec{j}$-parallel vector, from $\vec{u}=\langle a,b\rangle$
$=a\vec{i}+b\vec{j}$.



The $\vec{i}$-parallel vector $a\vec{i}$ is called the projection of $\vec{u}$ onto $\vec{i}$. This is like the shadow of $\vec{u}$ on the line of $\vec{i}$.



The projection of $\vec{u}$ onto $\vec{v}$ is denoted $\text{proj}_{\vec{v}}\vec{u}$,

and is given by

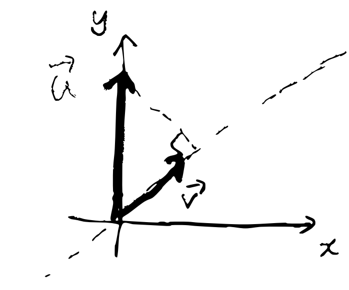
$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} \quad \leftarrow \text{vector}$$

(compare with)

$$\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} \quad \leftarrow \text{scalar.}$$

<u>Example</u> Let's consider projecting $\vec{u} = \langle 0,3\rangle$ onto

$\vec{v} = \langle 1,1\rangle$.



$\vec{u}\cdot\vec{v} = \langle 0,3\rangle\cdot\langle 1,1\rangle$

$= 3$

$|\vec{v}| = \sqrt{1^2+1^2} = \sqrt{2}$

$$\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = \frac{3}{\sqrt{2}}$$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{3}{(\sqrt{2})^2}\langle 1,1\rangle = \frac{3}{2}\langle 1,1\rangle = \langle \frac{3}{2},\frac{3}{2}\rangle = \langle 1.5, 1.5\rangle$$

<u>Example</u> For $\vec{u} = \langle 3, -1, 1 \rangle$, $\vec{v} = \langle 4, 7, -4 \rangle$,

$$\vec{u} \cdot \vec{v} = \langle 3, -1, 1 \rangle \cdot \langle 4, 7, -4 \rangle$$

$$= 3 \cdot 4 + (-1) \cdot 7 + 1 \cdot (-4) = 12 - 7 - 4 = 1$$

$$|\vec{v}| = \sqrt{4^2 + 7^2 + (-4)^2} = \sqrt{16 + 49 + 16}$$

$$= \sqrt{81} = 9$$

$$\text{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|} = \frac{1}{9}$$

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} = \frac{1}{9^2} \langle 4, 7, -4 \rangle = \frac{1}{81} \langle 4, 7, -4 \rangle$$

$$= \left\langle \frac{4}{81}, \frac{7}{81}, -\frac{4}{81} \right\rangle.$$